



TITLE:

Optimal Selection of the Penultimate Candidate (Mathematics of Decision Making under Uncertainty and Related Topics)

AUTHOR(S):

Tamaki, Mitsushi

CITATION:

Tamaki, Mitsushi. Optimal Selection of the Penultimate Candidate (Mathematics of Decision Making under Uncertainty and Related Topics). 数理解析研究所講究録 2019, 2126: 138-145

ISSUE DATE:

2019-08

URL:

<http://hdl.handle.net/2433/252243>

RIGHT:

Optimal Selection of the Penultimate Candidate

Mitsushi Tamaki

Faculty of Business Administration, Aichi University

Abstract

A known number n of objects appear one at a time. Let $X_k, 1 \leq k \leq n$, denote the value of the k th object and suppose that X_1, X_2, \dots, X_n are independent and identically distributed continuous random variables with a known distribution function. Let $L_k = \max(X_1, \dots, X_k)$, and call the k th object a *candidate* if it is a relative maximum, i.e. $X_k = L_k$. We denote by C_j the j th to last candidate, $j \geq 1$. Hence C_1 is the last candidate and C_2 the penultimate candidate, etc. The problem we consider here seeks a stopping rule that maximizes the probability of choosing C_2 . We give the optimal rule and the corresponding success probability. It can be shown that this success probability tends to 0.416002 as $n \rightarrow \infty$. Some comparisons with other related problems are also made.

1 Introduction

We first review the 'full-information' best-choice problem originally studied by Gilbert and Mosteller (1966, Sec.3) as a variation of the secretary problem. A known number n of objects appear one at a time. Let $X_k, 1 \leq k \leq n$, denote the value of the k th object and suppose that X_1, X_2, \dots, X_n are independent and identically distributed continuous random variables with a known distribution function F . As each object appears, we observe its value and decide either to select or reject it based on the values observed so far. The objective is to find a stopping rule that maximizes the probability of choosing the best, i.e. stopping with the largest of X_1, X_2, \dots, X_n , and compute the probability of choosing the best. We can assume without loss of generality that X_1, X_2, \dots, X_n are uniformly distributed on the interval $(0, 1)$, because order relationship is preserved under transformation $F(X_k)$.

Let $L_k = \max(X_1, \dots, X_k), 1 \leq k \leq n$, and call the k th object (or X_k) a *candidate* if it is a relative maximum, that is, $X_k = L_k$. A candidate is sometimes referred to as a record in the literature. Denote by C_j the j th to last candidate, $j \geq 1$. Hence C_1 is the last candidate (i.e. best overall) and C_2 the penultimate candidate, etc. Then we are tempted to consider

the problem of choosing C_j if it exists. Since the problem of choosing C_1 is just the best-choice problem, we here try to solve the problem of choosing C_2 as a first step toward these problems. Needless to say, this problem can be viewed as a two-choice problem of choosing both C_1 and C_2 , because the identification of C_2 also identifies C_1 as the candidate next to C_2 with certainty. The main results will be summarized in Section 2. In the 'no-information' analogue, where we can only observe the relative rank of the current object with respect to its predecessors, Bruss and Paindaveine (2000) solved the problem of choosing C_j for all $j(\geq 1)$. In Section 3, we compare our problem with other related problems.

2 Main results

The objective of the problem we consider here is to find a stopping rule that maximizes the probability of choosing C_2 , if any, and derive the corresponding success probability (if C_1 appears at the first stage, we do not have C_2 , so, in such a case, our trial is unsuccessful). Our problem makes sense for $n \geq 2$ and the decision of selection takes place only when a candidate appears.

For $n = 2$, the optimal rule obviously stops with X_1 and yields the success probability $P_2^* = P\{X_1 < X_2\} = 1/2$. For $n = 3$, the optimal rule also stops with X_1 and yields the success probability $P_3^* = 1/2$. This can be seen as follows. Suppose that we stop with X_1 . We then have three cases with respect to the size of X_1 relative to X_2 and X_3 . Since the respective success probabilities are $1/2$, 1 , 0 , depending on whether X_1 is the smallest, middle or largest, and these three cases are equally likely, the success probability is $(1/2)(1/3) + (1)(1/3) + (0)(1/3) = 1/2$. On the other hand, if we pass over X_1 , we succeed only when $X_1 < X_2 < X_3$, implying that the success probability is $1/6$. Thus the maximal success probability is $P_3^* = \max\{1/2, 1/6\} = 1/2$.

Before considering our problem for $n \geq 4$, we need some preparations. We review the distribution of the number of relative maxima. Let $r_m(k)$ be the probability that the total number of relative maxima is k when we observe m independent and identically distributed continuous random variables (or, equivalently, we observe the random permutation of m rankable objects). Then it is well known (see, e.g. Section 6.2 of Blom et al.(1994)) that $r_m(k)$ satisfies the recursion

$$r_m(k) = \frac{1}{m}r_{m-1}(k-1) + \left(1 - \frac{1}{m}\right)r_{m-1}(k), \quad 1 \leq k \leq m, \quad 2 \leq m,$$

with $r_1(1) = 1$ and $r_m(k) = 0$ for $k = 0$ or $k > m$. In particular, we have by induction

$$r_m(1) = \frac{1}{m}, \quad r_m(2) = \frac{h_{m-1}}{m} \tag{1}$$

where $h_j = \sum_{i=1}^j 1/i$ for $j \geq 1$.

We denote by (k, x) a state of the process where we have just observed the k th object to be a candidate having value x , i.e. $X_k = L_k = x$. Let $p_k(x)$ be the success probability by stopping immediately with the current candidate in state (k, x) and $q_k(x)$ be the one by stopping with the next candidate (after leaving (k, x)), if any. These two quantities are expressed as

$$p_k(x) = P\{M_k(x) = 1\} \quad (2)$$

$$q_k(x) = P\{M_k(x) = 2\}, \quad (3)$$

if we denote by $M_k(x)$ the number of future candidates. For further computation, we proceed by conditioning on $N_k(x)$ which denotes the number of future observations that are bigger than x . Since $N_k(x)$ is a binomial random variable with parameters $(n - k, 1 - x)$ and the conditional probability $P\{M_k(x) = 1 \mid N_k(x) = m\}$ is given as $r_m(1)$ from the former half of (1) regardless of x , we have from (2)

$$\begin{aligned} p_k(x) &= \sum_{m=1}^{n-k} P\{M_k(x) = 1 \mid N_k(x) = m\} P\{N_k(x) = m\} \\ &= \sum_{m=1}^{n-k} \frac{1}{m} \binom{n-k}{m} (1-x)^m x^{n-k-m} \end{aligned}$$

for $k \leq n - 1$ with $p_n(x) = 0$. In a similar manner, we have from (3) combined with the latter half of (1)

$$q_k(x) = \sum_{m=2}^{n-k} \frac{h_{m-1}}{m} \binom{n-k}{m} (1-x)^m x^{n-k-m}$$

for $k \leq n - 2$ with $q_{n-1}(x) = q_n(x) = 0$.

We are now ready to state the main results for $n \geq 4$.

Theorem 1. (a) *Optimal stopping rule:* Let, for $0 < x < 1$,

$$\phi_r(x) = \sum_{m=2}^r \frac{h_m - 1}{(m+1)^2} \binom{r}{m} \left(\frac{1-x}{x}\right)^m, \quad r \geq 2.$$

Then there exists an increasing sequence of the thresholds $\{b_r, r \geq 2\}$ with b_r defined as a unique solution $x \in (0, 1)$ to the equation

$$\phi_r(x) = 1 \quad (4)$$

such that the optimal rule is to choose the first candidate $X_k (= L_k)$ that exceeds the threshold $b_{(n-1)-k}$ (note that $(n-1) - k$ denotes the remaining

number of observations to the penultimate stage $n - 1$).

(b) *Maximal success probability P_n^* :* We have that

$$P_n^* = \int_{a_1}^1 p_1(x) dx + \sum_{k=2}^{n-1} \sum_{j=1}^{k-1} \int_{a_k}^1 p_k(x) \frac{[\min(x, a_j)]^{k-1}}{k-1} dx, \quad (5)$$

where $a_k = b_{(n-1)-k}$ for $1 \leq k \leq n-3$ and $a_k = 0$, otherwise.

(c) *Asymptotics:* Define $c^* (\approx 3.27201)$ as a unique solution c to the equation

$$\sum_{m=2}^{\infty} \frac{h_m - 1}{(m+1)^2} \cdot \frac{c^m}{m!} = 1. \quad (6)$$

Then, using the exponential-integral functions

$$\begin{aligned} I(c) &= \int_1^{\infty} \frac{e^{-cx}}{x} dx \\ J(c) &= \int_0^1 \frac{e^{cx} - 1}{x} dx = \sum_{j=1}^{\infty} \frac{c^j}{j!j}, \end{aligned}$$

we have that, as $n \rightarrow \infty$,

$$\begin{aligned} P_n^* \rightarrow P^* &= e^{-c^*} J(c^*) - \left\{ (1 + c^*) J(c^*) + e^{c^*} J(-c^*) \right\} I(c^*) \\ &\approx 0.416002. \end{aligned}$$

Proof. Omitted.

We conclude this section with some comments concerning the approximation for b_r and the simplification of P_n^* . It is easy to see from (4) that $b_2 = (3\sqrt{2} - 1)/17 \approx 0.1907$ and $b_3 \approx 0.3409$ is a unique root $x(>0)$ of the equation $85x^3 + 17x^2 - x - 5 = 0$. For large r , we can give an asymptotic approximation for the b_r . Write b_r as $1 - c_r/r$ and write (4) in the form

$$\sum_{m=2}^r \frac{h_m - 1}{(m+1)^2 m!} \prod_{k=0}^{m-1} \left[\left(1 - \frac{k}{r} \right) \left(\frac{c_r}{1 - c_r/r} \right) \right] = 1.$$

In order for this to stay equal to 1 as $r \rightarrow \infty$, the c_r must converge to a constant $c_r \rightarrow c^*$, where c^* satisfies the equation (6). Thus $b_r \approx 1 - 3.272/r$.

Performing integrations in (5) for $n = 4$, we have

$$P_4^* = \frac{11}{24} + \frac{1}{6}a_1 - \frac{1}{4}a_1^2 - \frac{5}{6}a_1^3 + \frac{17}{24}a_1^4 \approx 0.47618$$

because of $a_1 = b_2$. In a similar manner, we obtain another expression of P_n^* for general n , as given in Corollary 1 below. Consider a binomial random variable with parameters (r, p) and denote its cumulative distribution function by

$$B_{in}(m; r, p) = \sum_{k=0}^m \binom{r}{k} p^k (1-p)^{r-k}, \quad 0 \leq m \leq r.$$

Then, applying to (5) the well-known formula

$$\int_0^a x^{s-1} (1-x)^{t-1} dx = B(s, t) B_{in}(t-1; s+t-1, 1-a),$$

where $0 \leq a \leq 1$ and $B(s, t)$ is the *beta function* defined as

$$B(s, t) = \int_0^1 x^{s-1} (1-x)^{t-1} dx = \frac{(s-1)!(t-1)!}{(s+t-1)!}$$

for positive integers s and t , we obtain the following result.

Corollary 1. *We have*

$$P_n^* = A_n + \sum_{k=2}^{n-1} \sum_{j=1}^{k-1} [B_n(j, k) + C_n(j, k)],$$

where

$$\begin{aligned} A_n &= \sum_{m=1}^{n-1} \frac{1}{m} \binom{n-1}{m} B(m+1, n-m) [1 - B_{in}(m; n, 1-a_1)] \\ B_n(j, k) &= \frac{1}{k-1} \sum_{m=1}^{n-k} \frac{1}{m} \binom{n-k}{m} B(m+1, n-m) \\ &\quad \times [B_{in}(m; n, 1-a_j) - B_{in}(m; n, 1-a_k)] \\ C_n(j, k) &= \frac{a_j^{k-1}}{k-1} \sum_{m=1}^{n-k} \frac{1}{m} \binom{n-k}{m} B(m+1, n-m-k+1) \\ &\quad \times [1 - B_{in}(m; n-k+1, 1-a_j)]. \end{aligned}$$

3 Some comparisons

(a). In the no-information problem of choosing C_2 , Bruss and Paindaveine (2000) showed that there exists a threshold r_n such that the optimal rule stops with the first candidate that appears after time r_n , if any. Moreover, as $n \rightarrow \infty$, r_n/n tends to e^{-2} and the corresponding success probability tends to $2e^{-2} \approx 0.2707$ (compare this with our value $P^* \approx 0.4160$). It is also interesting to observe that the same limiting values appear in two

quite different optimal stopping problems; one is the best-choice problem with an unknown random number of objects having a uniform distribution on $\{1, \dots, n\}$ considered by Presman and Sonin (1972) and the other the *duration problem* considered in Section 2.2 of Ferguson et al. (1992), where the objective is to maximize the time of possession of a candidate.

(b). In (a), we pointed out the coincidences among three problems in the no-information case. How about the corresponding triple coincidences in the full-information case? The answer is "No". Samuels (2004) showed that the best-choice problem with uniform number of objects and the duration problem have the same asymptotic optimal payoff

$$\hat{P} = e^{-\hat{c}}J(\hat{c}) + \left\{e^{\hat{c}} - 1 - \hat{c}J(\hat{c})\right\}I(\hat{c}) \approx 0.43517,$$

where $\hat{c} \approx 2.1198$ is a unique solution c to the equation

$$-J(-c)e^c - J(c) = e^c - 1.$$

Our payoff P^* is obviously different from \hat{P} . See also Porosinski (1987, 2002), Petrucci (1980), Gneden (2004) and Mazalov and Tamaki (2006) for \hat{P} .

(c). Finally we compare our problem with two related problems. One is the best-choice problem. Its asymptotic optimal payoff is

$$P_1 = e^{-c_1} + (e^{c_1} - c_1 - 1)I(c_1) \approx 0.58016,$$

where $c_1 \approx 0.80435$ is a unique solution c to the equation $J(c) = 1$. For these, see Gilbert and Mosteller (1966), Samuels (1982), (1991), (2004), Gneden (2004) and Berezovsky and Gneden (1984). The other is a one-choice problem of choosing either C_1 or C_2 , considered by Tamaki (2010) (see Theorem 4.1 and Table 2 for $m = 2$). The asymptotic optimal payoff is given by

$$\begin{aligned} P_2 &= [e^{c_2}\{1 - J(-c_2)\} - (1 + c_2)\{1 + J(c_2)\}]I(c_2) \\ &\quad + e^{-c_2}\{1 + J(c_2)\} \\ &\approx 0.8424, \end{aligned}$$

where $c_2 \approx 1.5151$ is a unique solution c to the equation

$$\int_0^c \frac{-J(-x)e^x - J(x)}{x} dx = 1.$$

Note that P_2 here is denoted by P_2^* in Tamaki (2010) and that (7) follows since the functions $J_2(t)$ and $K_2(t)$ in Tamaki (2010) can be expressed as

$$\begin{aligned} J_2(t) &= 1 + J(t) \\ K_2(t) &= e^t - 1 + \left\{-J(-t)e^t - J(t)\right\} \end{aligned}$$

in terms of $J(\cdot)$. To derive (8) and (9), we have used Gnedin's identity (2004, p.322). Incidentally the asymptotic optimal payoff of the corresponding no-information problem is $(1 + \sqrt{2})e^{-\sqrt{2}} \approx 0.5869$ (see Tamaki (2010)).

(d). Let T_j be the random arrival time of C_j and Y_j be its random value on the PPP defined in the proof (c) of Theorem 1. Then random atom of C_j is identified as (T_j, Y_j) , $j \geq 1$. From the property of PPP, we easily find

$$\begin{aligned} T_j &= U_1 U_2 \cdots U_j \\ Y_j &= E_1 + \frac{E_2}{U_1} + \cdots + \frac{E_j}{U_1 U_2 \cdots U_{j-1}}, \end{aligned}$$

where U_1, U_2, \dots, U_j and E_1, E_2, \dots, E_j are all independent and U_k are uniform on $(0, 1)$ and E_k are exponential with parameter 1. For the best-choice problem, Samuels (2004) showed that, in his Section 10.3, P_1 has another expression

$$P_1 = P \left\{ Y_1 < \frac{c_1}{1 - T_1} \text{ and } Y_2 > \frac{c_1}{1 - T_2} \right\},$$

because we succeed if C_1 is below the optimal threshold $y = c_1/(1 - t)$ and C_2 is above. Similarly we have, as another expression for P^* ,

$$P^* = P \left\{ Y_2 < \frac{c^*}{1 - T_2} \text{ and } Y_3 > \frac{c^*}{1 - T_3} \right\},$$

because we succeed if C_2 is below the optimal threshold $y = c^*/(1 - t)$ and C_3 is above. This is equivalently written as

$$P^* = P \left\{ \left(E_1 + \frac{E_2}{U_1} \right) (1 - U_1 U_2) < c^* < \left(E_1 + \frac{E_2}{U_1} + \frac{E_3}{U_1 U_2} \right) (1 - U_1 U_2 U_3) \right\}.$$

See also Gnedin (2004) for EU-representation.

References

- [1] Berezovskiy, B. A. and Gnedin, A. V. (1984), *The Problem of Best Choice* (in Russian), Nauka, Moscow.
- [2] Blom, G., Holst, L. and Sandel, D. (1994). *Problems and Snapshots from the World of Probability*, Springer-Verlag, New York.
- [3] Bruss, F. T. and Paindaveine, D. (2000). Selecting a sequence of last successes in independent trials, *J. Appl. Prob.* **37**, 389-399.
- [4] Chow, Y. S., Robbins, H. and Siegmund, D. (1971). *Great Expectations: The Theory of Optimal Stopping*, Houghton-Mifflin, Boston.

- [5] Ferguson, T. S., Hardwick, J. P. and Tamaki, M. (1992). Maximizing the duration of owning a relatively best object, *Contemp. Math.* **125**, 37-57.
- [6] Ferguson, T. S. (2006). *Optimal Stopping and Applications*, Electronic Text at <http://www.math.ucla.edu/~tom/Stopping/Contents.html>
- [7] Gilbert, J.P. and Mosteller, F. (1966). Recognizing the maximum of a sequence, *J. Amer. Statist. Assoc.* **61**, 35-73.
- [8] Gnedin, A. V. (1996). On the full information best choice problem, *J. Appl. Prob.*, **33**, 678-687.
- [9] Gnedin, A.V. (2004). Best choice from the planar Poisson process, *Stoch. Process. Appl.* **111**, 317-354.
- [10] Mazalov, V. V. and Tamaki, M. (2006). An explicit formula for the optimal gain in the full-information problem of owning a relatively best object, *J. Appl. Prob.* **43**, 87-101.
- [11] Porosinski, Z. (1987), The full-information best-choice problem with a random number of observations, *Stoch. Process. Appl.* **24**, 293-307.
- [12] Porosinski, Z. (2002), On best choice problems having similar solutions, *Statist. Prob. Lett.* **56**, 321-327.
- [13] Presman, E. L. and Sonin, I. M. (1972). The best choice problem for a random number of objects, *Theor. Prob. Appl.* **17**, 657-668.
- [14] Samuels, S. M. (1982), Exact solutions for the full information best choice problem, *Purdue Univ. Stat. Dept. Mimeo. Series 82-17*.
- [15] Samuels, S. M. (1991), Secretary problems, *Handbook of Sequential Analysis* In B.K. Gosh and P.K. Sen(Eds.), Marcel Dekker, 381-405.
- [16] Samuels, S. M. (2004). Why do these quite different best-choice problems have the same solutions ? *Adv. Appl. Prob.* **36**, 398-416.
- [17] Tamaki, M. (2010). Sum the multiplicative odds to one and stop, *J. Appl. Prob.* **47**, 761-777.
- [18] Tamaki, M. (2015). On the optimal stopping problems with monotone thresholds, *J. Appl. Prob.* **52**, 926-940.

Postal address: Department of Business Administration, Aichi University, Nagoya Campus, Hiraike 4-60-6, Nakamura, Nagoya, Aichi 453-8777, Japan.

Email address: tamaki@vega.aichi-u.ac.jp

愛知大学・経営学部 玉置 光司